Stretched Exponential Relaxation in the Biased Random Voter Model

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Abstract

We study the relaxation properties of the voter model with i.i.d. random bias. We prove under mild condions that the disorder-averaged relaxation of this biased random voter model is faster than a stretched exponential with exponent $d/(d + \alpha)$, where $0 < \alpha \leq 2$ depends on the transition rates of the non-biased voter model. Under an additional assumption, we show that the above upper bound is optimal. The main ingredient of our proof is a result of Donsker and Varadhan (1979).

Keywords: biased voter model, annealed field, non-exponential relaxation, random walk.

1 Introduction

Recently, important progress has been made in the rigorous analysis of the dynamical properties of random spin systems. The best known class of random spin systems consists of spin glasses equiped with a Glauber dynamics (kinetic Ising models with random interaction and single spin-flip time evolution). These stochastic models are used in Monte Carlo simulations to mimic the time evolution of disordered systems, see e.g. Ogielski (1985).

The following results are stated with probability one with respect to the disorder and uniform in the initial condition. The assumptions under which they are derived differ slightly (cf. the original papers for the precise statements). Zegarlinski (1994) proves the absence of a spectral gap in random Glauber models in dimension $d \geq 1$. As a consequence, the decay of time dependent correlations in equilibrium cannot be exponentially fast. Moreover, he shows in dimension $d = 1$ that the decay of local functions can be bounded above by a stretched exponential whose exponent can be chosen arbitrarily in the open interval $(0, 1)$. Guionnet and Zegarlinski (1996) extend Zegarlinski's result. In particular, they find the same stretched exponential bound in dimension $d = 2$, even for continuous-spin systems. For $d \geq 3$ they are able to derive a relaxation faster than algebraic (Guionnet and Zegarlinski, 1997). Cesi, Maes and Martinelli (1997a) succeed in establishing for the Glauber model in arbitrary dimension $d \geq 1$, but under more restricted conditions on the distribution of the interaction, an upper bound for the asymptotic relaxation of local observables with decay faster than any stretched exponential. Furthermore, they obtain under even more severe assumptions (not valid for the diluted Ising model) a lower bound

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which is similar to the upper bound, implying that the latter cannot be improved in general.

Cesi, Maes and Martinelli (1997a) study also the (physically more relevant) disorder-averaged relaxation of Glauber models. In dimension $d \geq 2$ they find an upper bound with decay slower than any stretched exponential but faster than algebraic. In a particular case, they derive a lower bound which is again similar to the upper bound. Both bounds are in agreement with a non-rigorous result for the asymptotic relaxation of the spin autocorrelation function in the diluted Ising model below the treshold for bond percolation; see Bray (1989), and references therein.

Alexander, Cesi, Chayes, Maes and Martinelli (1998) confirm for the diluted Ising model in dimension $d \geq 2$ all the results of Cesi, Maes and Martinelli (1997a), even above the percolation treshold, except the almost sure lower bound which they prove only with positive probability; see also Cesi, Maes and Martinelli (1997b), and Martinelli (1997).

The previously mentioned results concern spin systems whose single-flip dynamics are reversible with respect to some Gibbs measure corresponding to a random potential. A powerful tool in the analysis of these systems is the so-called logarithmic Sobolev inequality. Gielis and Maes (1996) tackle the problem of estimating the asymptotic relaxation in a more general class of random spin systems with quenched disorder. Their method consists of a coupling of the spin system to the contact process and the use of percolation techniques on the graphical representation of the contact process. They obtain, with probability one with respect to the disorder, an upper bound which decays faster in time than any power law; see Remark 2.7 below. Moreover, in case of directed interactions they improve the upper bound to a stretched exponential whose exponent can be taken arbitrary in the open interval $(0, 1/2)$. No lower bound is known. However, it is believed that their estimate is far from optimal. In a forthcoming paper we will ameliorate their bound for the random voter model with quenched bias.

To the best of our knowledge, no rigorous bounds are yet available for the disorder-averaged relaxation of spin systems in this larger class. In the present paper we prove, under the mild assumption of a positive probability for each voter to be biased, that the disorder-averaged asymptotic relaxation of local functions in the biased random voter model is, uniform in the initial condition, bounded from above by a stretched exponential decay in time with exponent $d/(d + \alpha)$, where $d \geq 1$ is the dimension of the hypercubic lattice $\mathbb{Z}^d$ and $0 < \alpha \leq 2$ depends only on the transition rates of the non-biased voter model. Moreover, the upper bound cannot be improved in general: Under the assumption that each voter is non-biased with positive probability, we prove that the disorder-averaged asymptotic relaxation of monotone local functions is bounded from below by a stretched exponential with the same exponent. The approach of our proof consists of a coupling of the dual process of the biased voter model with a collection of independent random walks, and the application of a result of Donsker and Varadhan (1979) about the number of distinct sites visited by a random walk.

The model studied in this paper is an example of a spin system with an annealed random field. Often a distinction is made between spin systems with a random field and those with a random coupling between the constituents. The most famous example of the latter kind is the kinetic Ising model with an exchange interaction of random strength (as e.g. in the diluted Ising model).

We have restricted the above discussion to rigorous results for random spin systems. There are other particle systems in random environment worth mentioning. An extensively studied class of such systems consists of random walks in random environment; see e.g. den Hollander (1984), den Hollander, Naudts and Scheunders (1992), and den Hollander, Naudts and Redig (1992, 1994). The random trap model has even been studied with a time-evolving environment in den Hollander and Shuler (1992), and Redig (1994). The asymptotics of the survival probability in the random trap model is closely related to our result; see the discussion in den Hollander (1984), and the references given there.

The physics behind the phenomenon of slower than exponential decay is in all these cases the same. In the diffusion model with traps there exists large regions free of traps, be it with a small probability. Particles diffusing in trap-free regions will survive significantly longer than those in areas with average or high density of traps, contributing in an anomalous way to the asymptotics of the decay. Similarly, in spin glasses the ordering of the spins decays very slowly in areas of strong interaction. In the present model areas with a small amount of biased voters can display a dominancy of opinion 1 during a long time, before decaying to opinion 0.

This paper is organized as follows. In Section 2 we define the biased voter model and its randomized version, state our result, and mention the theorem of Donsker and Varadhan (1979) on which the proof of our result relies heavily. In Section 3 we describe the coalescing dual process of the biased voter model, and proof the upper and lower bound in Theorem 2.5.

2 Statement of result

A spin system is a continuous-time Markov process $(\eta_t)_{t \geq 0}$ with configuration space

$$\mathbb{X} = \{0,1\}^{\mathbb{Z}^d} = \{\eta : \mathbb{Z}^d \to \{0,1\} : x \mapsto \eta(x)\}. \tag{2.1}$$

Let η^x be the configuration obtained from configuration η by changing the state at site x, i.e.,

$$\eta^x(y) = \begin{cases} 1 - \eta(x) & \text{if } y = x, \\ \eta(y) & \text{if } y \neq x. \end{cases} \tag{2.2}$$

The infinitesimal dynamics of a spin system is specified by a Markov pregenerator Ω, given by

$$\Omega f(\eta) = \sum_{x \in \mathbb{Z}^d} c(x, \eta)[f(\eta^x) - f(\eta)] \tag{2.3}$$

and defined on a suitable subclass of $C(\mathbb{X})$, the set of real-valued continuous functions with domain $\mathbb{X}$. (On $\{0,1\}$ we assume the discrete topology and on $\mathbb{X}$ the product topology.) Such a suitable subclass is the collection of local functions defined below, see Liggett (1985). The non-negative (and uniformly bounded) quantities $c(x, \eta)$ in (2.3) are called the transition rates of the spin system. We start by specifying the transition rates of the biased (random) voter model.

Definition 2.1. The *biased voter model* is the spin system with transition rates

$$c_\beta(x, \eta) = \beta(x)\eta(x) + \sum_{y \in \mathbb{Z}^d} p(y - x)\{\eta(x)[1 - \eta(y)] + \eta(y)[1 - \eta(x)]\} \qquad (2.4)$$

where

$$\beta : \mathbb{Z}^d \to [0, \infty), \quad \sup_{x \in \mathbb{Z}^d} \beta(x) < \infty, \qquad (2.5a)$$

$$p : \mathbb{Z}^d \to [0, 1], \quad \sum_{x \in \mathbb{Z}^d} p(x) = 1, \quad \text{and} \quad p(0) = 0. \qquad (2.5b)$$

The *biased random voter model* is the spin system with transition rates (2.4) where $\{\beta(x), x \in \mathbb{Z}^d\}$ is a collection of independent identically distributed random variables with joint distribution $\mathbb{B}$.

The interpretation of the non-biased voter model (i.e., the spin system with transition rates (2.4), but with $\beta = 0$) is well known, see e.g. Liggett (1985). In this interpretation $\mathbb{Z}^d$ is viewed as a collection of individuals, each of which has one of two possible opinions (0 or 1) on a political issue. At the event times of a Poisson process the individual at site x reassesses his opinion $\eta(x)$ in the following way: he consults a "friend" y with probability $p(y - x)$ and then adopts his position $\eta(y)$. The presence of β in the transition rates of the biased voter model can be interpreted as the presence of a propaganda mechanism which drives each individual to opinion 0; $\beta(x)$ is then a measure for the susceptibility of individual x for that propaganda. Other interpretations of the biased (random) voter model are possible. E.g., it can be viewed as a zero-temperature kinetic Ising model in a (random) magnetic field.

From the explicit form of the transition rates it is immediate that configuration $\eta = 0$ is absorbing, and thus δ_0, the Dirac measure with unit mass at $\eta = 0$, is invariant. From the well-known ergodicity criterion $M < \varepsilon$ (Liggett, 1985) or from duality (see below) one can deduce that the biased voter model is (exponentially) ergodic whenever β is uniformly positive, i.e.

$$\inf_{x \in \mathbb{Z}^d} \beta(x) > 0, \qquad (2.6)$$

irrespective of the values of $p(x)$. On the other hand, the non-biased voter model ($\beta = 0$) is not ergodic. A more refined ergodicity criterion for the biased voter model should, of course, contain conditions on $p(x)$, e.g. irreducibility.

For the formulation of our main result we need the following definitions and assumption.

Definition 2.2. On $\mathbb{X}$ we define the partial order relation $\leq$ by

$$\eta \leq \zeta \Leftrightarrow \eta(x) \leq \zeta(x) \text{ for every } x \in \mathbb{Z}^d. \qquad (2.7)$$

A real-valued function $f : \mathbb{X} \to \mathbb{R}$ is said to be monotone whenever

$$\eta \leq \zeta \Rightarrow f(\eta) \leq f(\zeta). \qquad (2.8)$$

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A spin system with transition rates $c(x, \eta)$ is said to be attractive if

$$\eta \leq \zeta \Rightarrow \begin{cases} c(x, \eta) \leq c(x, \zeta) & \text{if } \eta(x) = \zeta(x) = 0 \\ c(x, \eta) \geq c(x, \zeta) & \text{if } \eta(x) = \zeta(x) = 1 \end{cases} \tag{2.9}$$

Thus, a spin system is attractive if, for every $x \in \mathbb{Z}^d$, $[1 - 2\eta(x)]c(x, \eta)$ is a monote function of η.

It is well known that a spin system is attractive if and only if it is monotone, in the sense that $S(t)f$, defined by $S(t)f(\eta) = \mathbb{E}^\eta f(\eta_t)$, is monotone if f is monotone (Liggett, 1985).

Definition 2.3. Let $\mathbb{Y} = \{A : A \text{ finite subset of } \mathbb{Z}^d\}$. This is a countable set. A function $f : \mathbb{X} \to \mathbb{R}$ is said to be local if there exists an $A \in \mathbb{Y}$ such that

$$\forall \eta, \zeta \in \mathbb{X} : \eta = \zeta \text{ on } A \Rightarrow f(\eta) = f(\zeta), \tag{2.10}$$

where $\eta = \zeta$ on A means that $\eta(x) = \zeta(x)$ for every $x \in A$. The smallest set A for which (2.10) holds will be denoted by $\Lambda(f)$. The space of continuous $C(\mathbb{X})$ will be equiped with the uniform norm $|| \cdot ||$, defined by

$$||f|| = \sup_{\eta \in \mathbb{X}} |f(\eta)|. \tag{2.11}$$

With respect to this norm, the set of local functions is dense in $C(\mathbb{X})$ (Stone–Weierstrass).

In the statement of Theorem 1, and Theorem 2, we will suppose that the following key assumption holds.

Assumption 2.4. Let, for every $k \in \mathbb{R}^d$,

$$\hat{p}(k) = \sum_{x \in \mathbb{Z}^d} p(x) \exp(i\langle k, x \rangle), \tag{2.12}$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product defined on $\mathbb{R}^d$. Assume that

- $\hat{p}(k) = 1$ if and only if $k = 2\pi(n_1, \dots, n_d)$, where $n_1, \dots, n_d$ are integers.

- $\hat{p}(k) = 1 - D(k) + o(|k|^\alpha)$ as $|k| = \sqrt{\langle k, k \rangle} \to 0$, where $0 < \alpha \leq 2$ and $\exp(-D(k))$ is the characteristic function of a symmetric stable law of index α in $\mathbb{R}^d$ which is non-degenerate, i.e.,

$$D(k) = \begin{cases} \displaystyle\sum_{i,j=1}^d D_{ij} k_i k_j \text{ for some positive definite matrix } D_{ij}, & \text{if } \alpha = 2, \\ \displaystyle\int_{\mathbb{S}^d} \int_0^\infty [1 - \cos\langle k, ry \rangle]\, r^{-(1+\alpha)}\, dr\, \sigma(dy), & \text{if } \alpha < 2. \end{cases} \tag{2.13}$$

In the expression for $D(k)$, σ is a symmetric measure on the unit ball $\mathbb{S}^d$ in $\mathbb{R}^d$, and the assumption of non-degeneracy means that the support of σ spans $\mathbb{S}^d$.

The above assumptions on $\hat{p}$ imply that the random walk with transition probabilities $p(x)$ is irreducible and the distribution $p(x)$ belongs to the domain of normal attraction of a non-degenerate symmetric stable law of index $0 < \alpha \le 2$ (Spitzer, 1976).

Theorem 2.5 (main result). *Suppose that Assumption 2.4 holds.*

(i) *If $\mathbb{B}(\{\beta : \beta(0) \ne 0\}) > 0$, then there exists a constant $\nu_1 > 0$ such that*

$$\limsup_{t \to \infty} t^{-d/(d+\alpha)} \log \int ||S_\beta(t)f - \delta_0(f)|| \, \mathbb{B}(d\beta) \le -C_{d,\alpha}(\nu_1) \tag{2.14a}$$

for every local function f.

(ii) *If $\mathbb{B}(\{\beta : \beta(0) = 0\}) > 0$, then there exists a constant $\nu_2 > \nu_1$ such that*

$$\liminf_{t \to \infty} t^{-d/(d+\alpha)} \log \int ||S_\beta(t)f - \delta_0(f)|| \, \mathbb{B}(d\beta) \ge -C_{d,\alpha}(\nu_2)|\Lambda(f)| \tag{2.14b}$$

for every non-constant monotone local function f.

In (2.14) $C_{d,\alpha}$ is an increasing function defined on $[0, \infty)$ given by

$$C_{d,\alpha}(\nu) = (d+\alpha)\left[\left(\frac{\lambda}{d}\right)^d \left(\frac{\nu}{\alpha}\right)^\alpha\right]^{1/(d+\alpha)} \tag{2.15}$$

where $\lambda > 0$ is a constant specified in Theorem 2.8 below.

Remark 2.6. In the proof of Theorem 2.5 we will choose the following values for ν_1 and ν_2:

$$\begin{aligned} \nu_1 &= -\log \int \frac{1}{1 + \beta(0)} \, \mathbb{B}(d\beta) \\ \nu_2 &= -\log \mathbb{B}(\{\beta : \beta(0) = 0\}) \end{aligned} \tag{2.16}$$

Using Jensen's inequality one sees easily that $\nu_1 \le \log(1 + \bar{\beta})$, where $\bar{\beta} = \int \beta(0) \, \mathbb{B}(d\beta)$.

Remark 2.7. The relaxation for typical realizations of the disorder in a general class of (quenched) random spin systems has been studied by Gielis and Maes (1996). Their result for the biased random voter model reads in our notations as follows; see also Klein (1994). *Suppose $p(x) = 0$ whenever $|x| \ne 1$, and let $K > \left(d + \sqrt{d(d+1)}\right)^2$. Then, there exists a constant $v_0 = v_0(K, d) > 1$ such that for all $1 < v < v_0$ and $m > 0$ there is a positive constant $\varepsilon = \varepsilon(K, d, m, v)$ such that*

$$\int \left[\log\left(1 + \frac{2}{\beta(0)}\right)\right]^K \mathbb{B}(d\beta) < \varepsilon \tag{2.17}$$

implies

$$\limsup_{t \to \infty} \frac{||S_\beta(t)f - \delta_0(f)||}{\left(\log(1 + t)\right)^v} \le -m \tag{2.18}$$

for every local function f and $\mathbb{B}$-almost every realization β of the disorder. Notice that a necessary condition for (2.17) to hold is $\mathbb{B}(\{\beta : \beta(0) = 0\}) = 0$.

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Theorem 2.5 will be proven in Section 3. The main ingredient of the proof is the following result of Donsker and Varadhan (1979). The restrictions we pose on the stochastic matrix $p(x, y)$ in Assumption 2.4 are sufficient for the validity of this result.

Theorem 2.8 (Donsker and Varadhan). *Let R_t be the colection of (distict) sites visited in the time interval $[0, t]$ by a continuous-time random walk $(X_t)_{t \geq 0}$ on $\mathbb{Z}^d$ with transition probabilities $\mathbb{P}^x(X_t = y) = p_t(y - x)$, where*

$$p_t(x) = e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} p^{(n)}(x) \tag{2.19}$$

and $p^{(n)}(x)$ are the n-step transition probabilities of the discrete-time random walk with one-step transition probabilities $p(x)$ (Spitzer, 1976). Suppose L is the infinitesimal generator of the symmetric stable process in $\mathbb{R}^d$ of index $0 < \alpha \leq 2$ corresponding to $D(k)$, i.e.,

$$Lf(x) = \begin{cases} \displaystyle\sum_{i,j=1}^{d} D_{ij} \frac{\partial^2}{\partial x_i \partial x_j} f(x) & \text{if } \alpha = 2 \\ \displaystyle\int_{\mathbb{S}^d} \int_0^{\infty} \left[\frac{f(x + ry) + f(x - ry)}{2} - f(x) \right] r^{-(1+\alpha)} \, dr \, \sigma(dy) & \text{if } \alpha < 2 \end{cases} \tag{2.20}$$

for every smooth real-valued function f on $\mathbb{R}^d$. Then, for every $\nu > 0$,

$$\lim_{t \to \infty} t^{-d/(d+\alpha)} \log \mathbb{E}^0 \exp(-\nu |R_t|) = -C_{d,\alpha}(\nu), \tag{2.21}$$

where $C_{d,\alpha}(\nu)$ is defined in Theorem 2.5 and

$$\lambda = \inf_G \lambda(G) > 0, \tag{2.22}$$

the infimum being over all open subsets G in $\mathbb{R}^d$ of unit volume and $\lambda(G)$ is the smallest eigenvalue of $-L$ with Dirichlet boundary conditions for G.

3 Proof of Theorem 2.5

The proof of Theorem 2.5 is an application of Theorem 2.8, combined with attractiveness and duality. We start with a description of the coalescing dual process of the biased voter model (see also Liggett, 1985).

3.1 Duality for the biased voter model

The dual process is a continuous-time Markov chain with configuration space $\mathbb{Y}$, see Definition 2.3 and the construction below. In case of coalescing duality the choice for the duality function is

$$H : \mathbb{X} \times \mathbb{Y} \to \{0, 1\} : (\eta, A) \mapsto \prod_{x \in A} \eta(x) \tag{3.1}$$

In the proof of Theorem 2.5 we will make use of the property that $H(\eta, A)$ is increasing in the first argument and decreasing in the second. Applying operator Ω_β, defined in (2.3) and (2.4), to the local function $H(\cdot, A)$ gives, for $A \in \mathbb{Y}$,

$$
\begin{aligned}
\Omega_\beta H(\eta, A) &= \sum_{x \in \mathbb{Z}^d} c_\beta(x, \eta)[H(\eta^x, A) - H(\eta, A)] \\
&= \sum_{B \in \mathbb{Y}} q(A, B)[H(\eta, B) - H(\eta, A)] - V_\beta(A)H(\eta, A)
\end{aligned}
\tag{3.2}
$$

where

$$
V_\beta(A) = \sum_{x \in A} \beta(x) \quad \text{(Feynman–Kac potential)}
\tag{3.3}
$$

and

$$
q(A, B) = \begin{cases}
\sum_{y \in A} p(y - x) & \text{if } \exists\, x \in A : B = A \setminus \{x\} \\
p(y - x) & \text{if } \exists\, x \in A, \exists\, y \notin A : B = (A \setminus \{x\}) \cup \{y\} \\
0 & \text{otherwise}
\end{cases}
\tag{3.4}
$$

The $q(A, B)$ are the transition rates of a continuous-time Markov chain $(A_t)_{t \geq 0}$ on $\mathbb{Y}$, which is called the coalescing dual process of the biased voter model. This terminology finds its origin in the interpretation of A_t as the finite collection of occupied sites in $\mathbb{Z}^d$ at time t; each site is occupied by at most one particle and only a finite number of particles are present. The transition rates $q(A, B)$ show that a particle at site $x \in \mathbb{Z}^d$ is removed after an exponentially distributed waiting time with mean 1, independent of the attendance of particles at other sites, and replaced with probability $p(y - x)$ by a particle at site $y \in \mathbb{Z}^d$. If after this action two particles occupy the same site y, these two particles coalesce.

Notice that the transition rates $q(A, B)$ are independent of the bias β and, in particular, the coalescing dual processes of the biased and the non-biased voter model coincide.

The following duality relation (or Feynman–Kac formula) explains why $(A_t)_{t \geq 0}$ is called a dual process of the biased voter model:

$$
\mathbb{E}^\eta_\beta H(\eta_t, A) = \mathbb{E}^A H(\eta, A_t) \exp\left\{ -\int_0^t V_\beta(A_s)\, ds \right\}
\tag{3.5}
$$

A proof of equation (3.5) can be found in Liggett (1985). Remark that the bias introduces a Feynman–Kac term in (3.5).

In the proof of Theorem 2.5 we will use the property that every local function f has a unique representation of the form

$$
f(\eta) = \sum_{A \in \mathbb{Y}} \hat{f}(A)H(\eta, A).
\tag{3.6}
$$

Obviously, $\hat{f}(A) = 0$ unless $A \subset \Lambda(f)$.

3.2 Upper bound

In this section we use the notation

$$\Sigma(f) = \sum_{A \neq \emptyset} |\hat{f}(A)|, \tag{3.7}$$

where f is an arbitrary local function, and start with the following estimate:

$$
\begin{aligned}
||S_\beta(t)f - \delta_0(f)|| &\leq \sum_{A \neq \emptyset} |\hat{f}(A)| \sup_{\eta \in \mathbb{X}} \mathbb{E}_\beta^\eta H(\eta_t, A) \\
&= \sum_{A \neq \emptyset} |\hat{f}(A)| \mathbb{E}_\beta^{\eta=1} H(\eta_t, A) \\
&\leq \Sigma(f) \max_{x \in \Lambda(f)} \mathbb{E}_\beta^1 H(\eta_t, \{x\}) \\
&= \Sigma(f) \max_{x \in \Lambda(f)} \mathbb{E}^{\{x\}} \exp\left\{ -\int_0^t V_\beta(A_s)\, ds \right\} \\
&= \Sigma(f) \max_{x \in \Lambda(f)} \mathbb{E}^x \exp\left\{ -\int_0^t \beta(X_s)\, ds \right\}
\end{aligned} \tag{3.8}
$$

The first step in (3.8) follows from identity (3.6) and

$$\int H(\eta, A)\, \delta_0(d\eta) = \begin{cases} 1 & \text{if } A = \emptyset \\ 0 & \text{if } A \neq \emptyset \end{cases}. \tag{3.9}$$

The second step in (3.8) uses the attractiveness of the biased voter model and the property that the duality function is increasing in its first argument. The third step in (3.8) is a consequence of the fact that the duality function is decreasing in its second argument. The fourth step in (3.8) is a particular case of the duality relation (3.5), while the fifth step uses that the dual process $(A_t)_{t \geq 0}$ with initial configuration $\{x\}$ is identical to the random walk $(X_t)_{t \geq 0}$ starting from position x. Integrating (3.8) with respect to $\mathbb{B}$ we get

$$
\begin{aligned}
\int ||S_\beta(t)f - \delta_0(f)||\, \mathbb{B}(d\beta) &\leq \Sigma(f) \int \max_{x \in \Lambda(f)} \mathbb{E}^x \exp\left\{ -\int_0^t \beta(X_s)\, ds \right\} \mathbb{B}(d\beta) \\
&\leq \Sigma(f) \int \sum_{x \in \Lambda(f)} \mathbb{E}^x \exp\left\{ -\int_0^t \beta(X_s)\, ds \right\} \mathbb{B}(d\beta) \\
&= \Sigma(f)|\Lambda(f)| \, \mathbb{E}^0 \int \exp\left\{ -\int_0^t \beta(X_s)\, ds \right\} \mathbb{B}(d\beta)
\end{aligned} \tag{3.10}
$$

The last step in (3.10) uses Fubini's theorem and the translation invariance of $\mathbb{B}$. To conclude the proof of the upper bound in Theorem 2.5 we need a suitable upper bound for

$$\mathbb{E}^0 \int \exp\left\{ -\int_0^t \beta(X_s)\, ds \right\} \mathbb{B}(d\beta). \tag{3.11}$$

To proceed, we consider the discrete-time analogue $(\tilde{X}_n)_{n \in \mathbb{N}}$ of $(X_t)_{t \geq 0}$. Let $(N_t)_{t \geq 0}$ be the Poisson process with unit rate,

$$\tilde{l}_n(x) = \sum_{m=0}^{n} \delta(x, \tilde{X}_m) \quad \text{and} \quad \tilde{R}_n = \{x \in \mathbb{Z}^d : \tilde{X}_m = x \text{ for some } m = 0, \dots, n\}. \quad (3.12)$$

The quantity in (3.11) equals

$$\sum_{n=0}^{\infty} \mathbb{P}(N_t = n) \, \tilde{\mathbb{E}}^0 \prod_{x \in \tilde{R}_n} \left(\int \frac{1}{1 + \beta(0)} \, \mathbb{B}(d\beta) \right)^{\tilde{l}_n(x)} \quad (3.13)$$

Using $\tilde{l}_n(x) \geq 1$ when $x \in \tilde{R}_n$ shows that (3.13) is bounded above by

$$\sum_{n=0}^{\infty} \mathbb{P}(N_t = n) \, \tilde{\mathbb{E}}^0 \left(\int \frac{1}{1 + \beta(0)} \, \mathbb{B}(d\beta) \right)^{|\tilde{R}_n|} \quad (3.14)$$

Returning to the continuous-time random walk $(X_t)_{t \geq 0}$ we obtain

$$\mathbb{E}^0 \int \exp\left\{ -\int_0^t \beta(X_s) \, ds \right\} \mathbb{B}(d\beta) \leq \mathbb{E}^0 \left(\int \frac{1}{1 + \beta(0)} \, \mathbb{B}(d\beta) \right)^{|R_t|} \quad (3.15)$$

with R_t the range of the continuous-time random walk $(X_t)_{t \geq 0}$ in the time interval $[0, t]$, i.e.,

$$R_t = \{x \in \mathbb{Z}^d : X_s = x \text{ for some } 0 \leq s \leq t\}. \quad (3.16)$$

Defining ν_1 as in Remark 2.6 and applying Theorem 2.8, completes the proof of the first statement in Theorem 2.5. $\qquad \square$

3.3 Lower bound

The main purpose of the second statement of Theorem 2.5 is to prove that the upper bound in the first statement is optimal. In the proof of the second part of Theorem 2.5 we make use of the following two lemmas.

Lemma 3.1. *A local function* $f : \mathbb{X} \to \mathbb{R}$ *is monotone if and only if*

$$\sum_{\substack{A \subset B_2 \\ A \cap (B_2 \backslash B_1) \neq \emptyset}} \hat{f}(A) \geq 0 \quad \text{for every } B_1 \subset B_2 \subset \mathbb{Z}^d. \quad (3.17)$$

Proof. The conclusion of the lemma follows from the observation

$$f(\zeta) - f(\eta) = \sum_{A \neq \emptyset} \hat{f}(A)[H(\zeta, A) - H(\eta, A)] \quad (3.18)$$

and the monotonicity of $H(\cdot, A)$ for every $A \in \mathbb{Y}$, because $\eta \leq \zeta$ if and only if there exist two subsets of $\mathbb{Z}^d$, B_1 and B_2, such that $B_1 \subset B_2$, $\eta = 1$ on B_1, $\eta = 0$ off B_1, $\zeta = 1$ on B_2, and $\zeta = 0$ off B_2. $\qquad \square$

Lemma 3.2. *Suppose Λ is a finite set, $\{x_A, A \subset \Lambda\} \subset \mathbb{R}$, $\{y_A, A \subset \Lambda\} \subset [0,1]$, $x_\emptyset = 0$, $\sum_{B \subset A} x_B \geq 0$ and $y_A = \prod_{a \in A} y_{\{a\}}$ for every $A \subset \Lambda$, where it is understood that $y_\emptyset = 1$. Then, for $|\Lambda| = 2, 3, \ldots,$*

$$\sum_{A \subset \Lambda} x_A y_A (1 - y_{\Lambda \setminus A}) \geq \sum_{a \in \Lambda} x_{\{a\}} y_{\{a\}} \prod_{b \in \Lambda \setminus \{a\}} (1 - y_{\{b\}}) \geq 0. \tag{3.19}$$

In particular,

$$\sum_{A \subset \Lambda} x_A y_A \geq \left(\sum_{A \subset \Lambda} x_A \right) y_\Lambda. \tag{3.20}$$

The last inequality also holds if $|\Lambda| = 1$.

Proof. Let, for $A \subset \Lambda$,

$$z_A = \sum_{B \subset A} x_B. \tag{3.21}$$

Inverting (3.21) yields

$$\begin{aligned}
\sum_{A \subset \Lambda} x_A y_A (1 - y_{\Lambda \setminus A}) &= \sum_{A \subset \Lambda} \left(\sum_{B \subset A} (-1)^{|A \setminus B|} z_B \right) y_A (1 - y_{\Lambda \setminus A}) \\
&= \sum_{B \subset \Lambda} z_B \sum_{A : B \subset A \subset \Lambda} (-1)^{|A \setminus B|} y_A (1 - y_{\Lambda \setminus A}) \\
&= \sum_{B \subset \Lambda} z_B \sum_{D \subset \Lambda \setminus B} (-1)^{|D|} (y_{B \cup D} - y_{(\Lambda \setminus B) \cup B}) \\
&= \sum_{B \subset \Lambda} z_B y_B \sum_{D \subset \Lambda \setminus B} (-1)^{|D|} (y_D - y_{\Lambda \setminus B}) \\
&= \sum_{\substack{B \subset \Lambda \\ B \neq \emptyset}} z_B y_B \prod_{a \in \Lambda \setminus B} (1 - y_{\{a\}})
\end{aligned} \tag{3.22}$$

The last sum contains only non-negative terms. Thus, reducing it to a sum over all singletons in Λ gives a lower bound. Finally, remarking that $z_{\{a\}} = x_{\{a\}}$ for every $a \in \Lambda$, proofs (3.19). $\qquad\square$

Due to the attractiveness of the biased voter model, $S_\beta(t)f$ is monotone whenever f is monotone. In particular,

$$S_\beta(t)f - \delta_0(f) \geq 0 \tag{3.23}$$

when f is monotone (because δ_0 is invariant). Moreover, applying inequality (3.20) of Lemma 3.2 to $x_A = \hat{f}(A)$ (see also Lemma 3.1), $y_A = H(\eta_t, A)$, and $\Lambda = \Lambda(f)$, shows that

$$S_\beta(t)f(\eta) - \delta_0(f) = \sum_{A \neq \emptyset} \hat{f}(A) \mathbb{E}_\beta^\eta H(\eta_t, A) \geq \left(\sum_{A \neq \emptyset} \hat{f}(A) \right) \mathbb{E}_\beta^\eta H(\eta_t, \Lambda(f)) \geq 0 \tag{3.24}$$

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Due to attractiveness and the monotonicity of $H(\cdot, A)$,

$$||S_\beta(t)f - \delta_0(f)|| \geq \left(\sum_{A \neq \emptyset} \hat{f}(A)\right) \mathbb{E}^1_\beta H(\eta_t, \Lambda(f)) \tag{3.25}$$

Now, we will derive a lower bound for $\int \mathbb{E}^1_\beta H(\eta_t, \Lambda(f)) \, \mathbb{B}(d\beta)$. First notice that

$$
\begin{aligned}
\int \mathbb{E}^1_\beta H(\eta_t, A) \, \mathbb{B}(d\beta) &= \int \mathbb{E}^A \exp\left\{-\sum_{x \in \mathbb{Z}^d} \beta(x) l_t(x)\right\} \mathbb{B}(d\beta) \\
&= \mathbb{E}^A \prod_{x \in \bar{R}_t} \int \exp\{-\beta(x) l_t(x)\} \, \mathbb{B}(d\beta) \\
&\geq \mathbb{E}^A \prod_{x \in \bar{R}_t} \mathbb{B}(\{\beta : \beta(x) = 0\}) \\
&= \mathbb{E}^A \exp(-\nu_2 |\bar{R}_t|)
\end{aligned}
\tag{3.26}
$$

where ν_2 is defined in Remark 2.6,

$$l_t(x) = \int_0^t \mathbb{I}_{A_s}(x) \, ds \quad \text{and} \quad \bar{R}_t = \bigcup_{0 \leq s \leq t} A_s. \tag{3.27}$$

If $A = \{x^1, \dots, x^n\}$ with distict x^i's, we couple (A_t) and $(X_t^1, \dots, X_t^n)$ with $X_0^i = x^i$ in such a way that $A_t \subset \{X_t^1, \dots, X_t^n\}$ for all $t \geq 0$, where $X_t^1, \dots, X_t^n$ are independent random walks with transition probabilities $p_t(x)$ (identical to the coupling used in the proof of Lemma 1.5 of Chapter V in Liggett (1985)). Since in this coupling the range $\bar{R}_t$ of the process (A_t), see (3.27), can only be smaller than the range R_t^n of the process $(X_t^1, \dots, X_t^n)$, i.e.,

$$R_t^n = \bigcup_{0 \leq s \leq t} \{X_s^1, \dots, X_s^n\}, \tag{3.28}$$

we find

$$
\begin{aligned}
\mathbb{E}^A \exp(-\nu_2 |\bar{R}_t|) &\geq \mathbb{E}^{(x^1, \dots, x^n)} \exp(-\nu_2 |R_t^n|) \\
&\geq \prod_{x \in A} \mathbb{E}^x \exp(-\nu_2 |R_t|) \\
&= \left[\mathbb{E}^0 \exp(-\nu_2 |R_t|)\right]^{|A|}
\end{aligned}
\tag{3.29}
$$

In the two last lines of (3.29) R_t is defined by (3.16). Summarizing,

$$\log \int ||S_\beta(t)f - \delta_0(f)|| \, \mathbb{B}(d\beta) \geq \log \sum_{A \neq \emptyset} \hat{f}(A) + |\Lambda(f)| \log \mathbb{E}^0 \exp(-\nu_2 |R_t|). \tag{3.30}$$

Applying Theorem 2.8 gives the desired result. $\qquad\square$

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References

K. S. Alexander, F. Cesi, L. Chayes, C. Maes, and F. Martinelli, *Convergence to equilibrium of random Ising models in the Griffiths' phase*, J. Stat. Phys. **92**, 337–352 (1998).

A. J. Bray, *Upper and lower bounds on dynamic correlations in the Griffiths phase*, J. Phys. A **22**, L81–L85 (1989).

F. Cesi, C. Maes, and F. Martinelli, *Relaxation of disordered magnets in the Griffiths' regime*, Comm. Math. Phys. **188**, 135–173 (1997a).

F. Cesi, C. Maes, and F. Martinelli, *Relaxation to equilibrium for two dimensional disordered Ising systems in the Griffiths' phase*, Comm. Math. Phys. **189**, 323–335 (1997b).

M. D. Donsker and S. R. S. Varadhan, *On the number of distinct sites visited by a random walk*, Comm. Pure Appl. Math. **32**, 721–747 (1979).

G. Gielis and C. Maes, *Percolation techniques in disordered spin flip dynamics: relaxation to the unique invariant measure*, Comm. Math. Phys. **177**, 83–101 (1996).

A. Guionnet and B. Zegarlinski, *Decay to equilibrium in random spin systems on a lattice*, Comm. Math. Phys. **181**, 703–732 (1996).

A. Guionnet and B. Zegarlinski, *Decay to equilibrium in random spin systems on a lattice II*, J. Stat. Phys. **86**, 899–904 (1997).

W. Th. F. den Hollander, *Random walks on lattices with randomly distributed traps. I. The average number of steps until trapping*, J. Stat. Phys. **37**, 331–367 (1984).

F. den Hollander, J. Naudts, and F. Redig, *Invariance principle for the stochastic Lorentz lattice gas*, J. Stat. Phys. **66**, 1583–1598 (1992).

F. den Hollander, J. Naudts, and F. Redig, *Dynamic structure factor in a random diffusion model*, J. Stat. Phys. **76**, 1267–1286 (1994).

F. den Hollander, J. Naudts, and P. Scheunders, *A long-time tail for random walk in random scenery*, J. Stat. Phys. **66**, 1527–1555 (1992).

F. den Hollander and K. E. Shuler, *Random walks in a random field of decaying traps*, J. Stat. Phys. **67**, 13–31 (1992).

A. Klein, *Extinction of contact and percolation processes in a random environment*, Ann. Prob. **22**, 1227–1251 (1994).

T. M. Liggett, *Interacting Particle Systems* (Grundlehren der mathematischen Wissenschaften 273, Springer–Verlag, 1985).

F. Martinelli, *Lectures on Glauber dynamics for discrete spin models* (Proceedings of the

Saint Flour summer school in probability 1997, to appear in Lecture Notes in Mathematics, Springer–Verlag).

A. T. Ogielski, *Dynamics of three-dimensional Ising spin glasses in thermal equilibrium*, Phys. Rev. B **32**, 7384–7398 (1985).

F. Redig, *An exponential upper bound for the survival probability in a dynamic random trap model*, J. Stat. Phys. **74**, 815–828 (1994).

F. Spitzer, *Principles of Random Walk* (Graduate Texts in Mathematics 34, Springer–Verlag, 1976).

B. Zegarlinski, *Strong decay to equilibrium in one-dimensional random spin systems*, J. Stat. Phys. **77**, 717–732 (1994).